Exact solutions of the magnetohydrodynamic equations

By W. E. WILLIAMS
Department of Applied Mathematics, Liverpool University

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Exact one-dimensional solutions of the magnetohydrodynamic equations of an incompressible fluid are considered. It is shown that one class of plane wave solutions of the linearized equations is also a possible class of solutions of the general equations including the effect of displacement current. A similar result is also established for the solutions for a horizontally stratified fluid. For the particular case when the viscosity is equal to the magnetic diffusivity an exact solution is obtained for the magnetohydrodynamic Rayleigh problem for a semi-infinite plate. It is shown that this solution may be employed directly to give the solution for liquids of small, but not necessarily equal, viscosity and magnetic diffusivity.

1. Introduction

The solution of the linearized equations of magnetohydrodynamics is a topic which has received a considerable amount of attention in recent years. The most comprehensive and systematic work on this subject is that of Baños (1955*a*, *b*), who has considered the detailed form of plane wave solutions of the linearized equations. It has also been observed by certain authors (e.g. Lundquist 1952) that, for an incompressible fluid, the general non-linear equations also possess plane wave solutions similar to those of the linearized equations. It thus seems of interest to consider other conditions under which the general and linear equations of an incompressible fluid possess similar exact solutions.

In the present paper we examine whether there exist similar solutions of the linear and exact equations of motion of an incompressible fluid which are functions of one Cartesian variable z and time.

In § 2 we consider the particular class of solutions with no velocity component in the z-direction. If the displacement current is neglected the exact equations are identical with the linearized ones and thus exact solutions of the linear equations are also solutions of the complete equations. In particular the forms of the plane wave solutions of the linear and exact equations are identical. This fact has been observed by Lundquist and others. It is also shown that, without neglecting the displacement current, there exist solutions of the linear equations which are also solutions of the exact equations. For plane waves these are the pressure modes defined by Baños (1955*b*) and are characterized by the fact that the associated Poynting vector is in the direction of propagation.

In § 3 the particular case when the external magnetic field is in the z direction is considered. It is shown that, for a horizontally stratified incompressible fluid, the small amplitude Alfvén wave type solutions obtained by Ferraro (1954) represent possible exact solutions of the equations of motion with the effect of displacement current included. It is also shown that the equations of motion for one-dimensional disturbances in an unbounded fluid are essentially linear and capable of exact solution.

The type of solutions considered in §2 occur in some initial value problems in magnetohydrodynamics; one such problem is the Rayleigh problem for an infinite plate when there is a transverse field present perpendicular to the plate. The solution of this problem has been considered in detail by Ludford (1959) and Chang & Yen (1959). Another class of boundary-value problem satisfying our conditions is the problem of the motion of a viscous fluid bounded by an infinite oscillating plane, and the solution for an insulating plane has been given by Kakutani (1959). The magnetohydrodynamic Rayleigh problem for a half-plane is another case when the general equations are exactly reducible to a linear form. This problem is considered in §4 where it is shown that, when the viscosity is equal to the magnetic diffusivity (conductivity $\times$ permeability)$^{-1}$, the boundary-value problem reduces to a classical one. It is also shown that the first-order solution for small viscosity and diffusivity may be deduced immediately from this special case.

2. General equations

The motion of an incompressible conducting fluid under the action of a constant external magnetic field $\mathbf{H}_0$ is governed by the equations

$$\mu\frac{\partial \mathbf{H}}{\partial t} = -\operatorname{curl}\mathbf{E}, \tag{1}$$

$$\frac{1}{\sigma}(\mathbf{J}-\rho_e\mathbf{v}) = \mathbf{E}+\mu\mathbf{v}(\mathbf{H}_0+\mathbf{H}), \tag{2}$$

$$\rho\frac{d\mathbf{v}}{dt} = -\operatorname{grad}p+\rho_e\mathbf{E}+\mu\mathbf{J}\times(\mathbf{H}_0+\mathbf{H})+\lambda\nabla^2\mathbf{v}, \tag{3}$$

$$\mathbf{J} = \operatorname{curl}\mathbf{H}-\epsilon\frac{\partial \mathbf{E}}{\partial t}, \tag{4}$$

$$\rho_e = \epsilon\operatorname{div}\mathbf{E}, \tag{5}$$

$$\operatorname{div}\mathbf{v} = 0, \tag{6}$$

$$\frac{\partial\rho}{\partial t}+\mathbf{v}.\operatorname{grad}\rho = 0. \tag{7}$$

In the above equations $\mathbf{E}$ and $\mathbf{H}$ denote the induced electric and magnetic fields, $\mathbf{J}$ denotes the current density, $\mathbf{v}$ the fluid velocity and p, ρ and ρ_e the pressure, fluid density and charge density, respectively. It will also be assumed that μ (permeability), ϵ (dielectric constant), σ (conductivity) and λ (coefficient of viscosity) are constant. We also have from Maxwell's equation that

$$\operatorname{div}\mathbf{H} = 0. \tag{8}$$

We now investigate the possibility of obtaining exact solutions of the above equations in which the field components are functions of one Cartesian variable z and of time. Equations (1) and (8) then show that H_z is constant; it may thus be absorbed into the z-component of $\mathbf{H}_0$ and hence there will be no loss of generality in taking $H_z = 0$. Equations (6) and (7) have, as one solution,

$$v_z = \partial\rho/\partial t = \partial\rho/\partial z = 0$$

and we consider solutions of equations (1)–(5) satisfying these conditions. Equation (7) shows that ρ will be constant if constant at any particular instant.

For the first part of the investigation, it will be assumed that the displacement current may be neglected (i.e. $\rho_e = 0$).

The above assumptions enable (3) to be rewritten

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{k}\frac{\partial}{\partial z}[p+\tfrac{1}{2}\mu(\mathbf{H}_0+\mathbf{H})^2]+\lambda\frac{\partial^2\mathbf{v}}{\partial z^2}+\mu H_{0z}\frac{\partial \mathbf{H}}{\partial z}, \tag{9}$$

where $\mathbf{k}$ is the unit vector in the z-direction. Equations (1), (2), (4) and (8) now give

$$\frac{1}{\sigma}\operatorname{curl}\mathbf{J} = -\frac{1}{\sigma}\frac{\partial^2\mathbf{H}}{\partial z^2} = -\mu\frac{\partial\mathbf{H}}{\partial t}+\mu H_{0z}\frac{\partial\mathbf{v}}{\partial z}. \tag{10}$$

Since $v_z = H_z = 0$ we have, from (9), that

$$\frac{\partial}{\partial z}\{p+\tfrac{1}{2}\mu(\mathbf{H}_0+\mathbf{H})^2\} = 0. \tag{11}$$

$\mathbf{H}$ may now be eliminated from (9) and (10) to give

$$\left(\rho\frac{\partial}{\partial t}-\lambda\frac{\partial^2}{\partial z^2}\right)\left(\mu\frac{\partial}{\partial t}-\frac{1}{\sigma}\frac{\partial^2}{\partial z^2}\right)\mathbf{v} = \mu^2H_{0z}^2\frac{\partial^2\mathbf{v}}{\partial z^2}. \tag{12}$$

Equation (12) has been obtained by Ludford (1959) for the particular case when $\mathbf{v}$ and $\mathbf{H}_0$ are perpendicular, it has also been derived by Dungey (1958) by linearizing the equations of motion.

In view of the fact that our assumptions concerning the form of the solutions have reduced the equations to a linear form it seems of interest to examine whether there exist solutions of the linearized equations which satisfy our assumptions and which may therefore represent solutions of the general equations. Since z is arbitrary it is seen that plane waves satisfy our assumptions, and the propagation of small amplitude plane waves has been considered in detail by Baños (1955*a*, *b*). Baños's work on magnetohydrodynamic waves is confined to the case of an inviscid fluid; in a later investigation (Baños 1956) he has considered the propagation of magneto-elastic waves. With suitable changes of notation the latter work may be used to obtain the results for the propagation of small amplitude plane waves in a viscous fluid.

Baños has shown that the solution of the linearized equations can be split up into two independent classes of solutions which he calls v- and p-modes. The v-modes are solutions which are obtained by requiring that the velocity be perpendicular to both $\mathbf{k}$ and $\mathbf{H}_0$; the pressure associated with such modes is then shown to be zero. The p-mode solutions arise when the fluid velocity is in the plane of $\mathbf{k}$ and $\mathbf{H}_0$ and perpendicular to $\mathbf{k}$; in general the pressure associated with this mode is non-zero. Another difference between the v- and p-modes is that the Poynting vector for the v-modes is in the direction of $\mathbf{H}_0$ whilst that for the

p-modes is parallel to $\mathbf{k}$. If the displacement current is neglected the wave-numbers of the v- and p-modes are equal.

We now examine the extent to which Baños's results are applicable to plane wave solutions of the general equations. If it is assumed that $\mathbf{v}$ and $\mathbf{H}$ are both proportional to $\exp i(\omega t + kz)$, then equation (12) gives

$$(i\omega\rho + \lambda k^2)(i\omega\mu + k^2\sigma^{-1}) = -\mu^2 H_{0z}^2 k^2. \tag{13}$$

For $\lambda = 0$ equation (13) reduces to one obtained by Baños (1955*b*) and for non-zero λ may be identified with a similar equation derived by Baños (1956) for magneto-elastic waves. Thus, if the displacement current is neglected, the wave-numbers of the plane wave solutions of the exact and linearized equations are equal.

The existence of v- and p-mode types of solution of the general equations will now be considered. It is easily seen that equation (9) may be re-written

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{k}\frac{\partial}{\partial z}(p + \tfrac{1}{2}\mu H^2) + \lambda\frac{\partial^2 \mathbf{v}}{\partial z^2} + \mu \mathbf{J}\times\mathbf{H}_0. \tag{14}$$

Clearly a possible class of solutions of (14) is obtained by setting $p = -\frac{1}{2}\mu H^2$; $\mathbf{v}$ will then be perpendicular to $\mathbf{H}_0$. Hence, since $v_z = 0$, the velocity is perpendicular to both $\mathbf{k}$ and $\mathbf{H}_0$. This is precisely the linear $\mathbf{v}$-mode and hence we conclude that the exact equations, neglecting displacement currents, possess plane wave solutions of the v-mode type. In the exact solution, however, the pressure associated with this mode is $-\frac{1}{2}\mu H^2$ and not zero; in the linear theory terms of order H^2 are neglected and this would give $p = 0$. Equation (13) shows that there will be two possible v-mode solutions, one of which is essentially a damped Alfvén wave and the other a highly attenuated one which vanishes for zero viscosity or infinite conductivity. $\mathbf{v}$ will be of the form $A\mathbf{k}\times\mathbf{H}_0 \exp i(\omega t + kz)$ and $\mathbf{H}$ and $\mathbf{E}$ may be obtained from equations (1) and (10).

It would be logical at this point to consider the existence of p-mode type solutions. It will, however, be shown that in this case it is not necessary to neglect the displacement current and we therefore consider conditions under which the effect of the displacement current may be treated simply. The non-linear effect of the displacement current is due to the charge density term and thus, in order to seek simple solutions, we shall consider only those with $\operatorname{div}\mathbf{E} = 0$. Since our main interest lies in solutions of the plane wave type it may be assumed that $E_z = 0$. If the displacement current is included then (10) becomes

$$-\frac{1}{\sigma}\frac{\partial^2 \mathbf{H}}{\partial z^2} + \epsilon\mu\frac{\partial^2 \mathbf{H}}{\partial t^2} = \mu\frac{\partial \mathbf{H}}{\partial t} + \mu H_{0z}\frac{\partial \mathbf{v}}{\partial z}. \tag{15}$$

Equation (15) shows that $\mathbf{v}$ and $\mathbf{H}$ are parallel for plane waves, and equations (2) and (3) thus become

$$\frac{1}{\sigma}\mathbf{J} = \mathbf{E} + \mu\mathbf{v}\times\mathbf{H}_0 \tag{16}$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\operatorname{grad}(p + \tfrac{1}{2}\mu H^2) + \mu\mathbf{J}\times\mathbf{H}_0 + \lambda\frac{\partial^2 \mathbf{v}}{\partial z^2} - \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}\times\mathbf{H}. \tag{17}$$

The only non-vanishing component of the last term on the right-hand side of (17) is that in the z-direction and hence this term may be written as $-\epsilon\,\mathrm{grad}\,\psi$, where ψ is a scalar depending on z and t, and hence

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\mathrm{grad}(p+\tfrac{1}{2}\mu H^2+\epsilon\psi)+\lambda\frac{\partial^2 \mathbf{v}}{\partial z^2}+\mu\mathbf{J}\times\mathbf{H}_0. \tag{18}$$

Equations (2) and (4) give

$$\frac{1}{\sigma}\mathbf{J}.\mathbf{H}_0 = \mathbf{E}.\mathbf{H}_0, \tag{19}$$

$$\left(1+\frac{\epsilon}{\sigma}\frac{\partial}{\partial t}\right)\mathbf{J}.\mathbf{H}_0 = \mathbf{H}_0.\mathrm{curl}\,\mathbf{H}. \tag{20}$$

From (1) and (19)

$$\left\{\frac{\partial}{\partial t}\left(1+\frac{\epsilon}{\sigma}\frac{\partial}{\partial t}\right)-\frac{1}{\sigma\mu}\frac{\partial^2}{\partial z^2}\right\}\mathbf{J}.\mathbf{H}_0 = -\frac{H_{0z}}{\mu}\frac{\partial}{\partial z}(\mathrm{div}\,\mathbf{E}). \tag{21}$$

Thus, since $\mathrm{div}\,\mathbf{E} = 0$, $\mathbf{J}.\mathbf{H}_0 = \mathbf{E}.\mathbf{H}_0 = 0$ is a possible solution and $\mathbf{E}$ will have the form $\phi\mathbf{k}\times\mathbf{H}_0$ where ϕ is a scalar. From (16)

$$\left(\rho\frac{\partial}{\partial t}-\lambda\frac{\partial^2}{\partial z^2}\right)\mathrm{curl}\,\mathbf{v} = \mu\,\mathrm{curl}\,(\mathbf{J}\times\mathbf{H}_0), \tag{22}$$

and eliminating $\mathbf{v}$ between (15) and (21) gives

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial z^2}\right)\mathrm{curl}\left[\mathrm{curl}\left\{\frac{1}{\sigma}\frac{\partial \mathbf{J}}{\partial t}-\frac{\partial \mathbf{E}}{\partial t}\right\}\right] = \frac{\mu^2}{\rho}H_{0z}\frac{\partial^2}{\partial t\,\partial z}\mathrm{curl}\,(\mathbf{J}\times\mathbf{H}_0), \tag{23}$$

where ν is the kinematic viscosity.

Finally, from (1), (4) and (22),

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial z^2}\right)\left\{-\frac{1}{\sigma\mu}\left(\frac{\partial^2}{\partial z^2}-\epsilon\mu\frac{\partial^2}{\partial t^2}\right)+\frac{\partial}{\partial t}\right\}\frac{\partial^2\phi}{\partial z^2} = \frac{\mu H_{0z}^2}{\rho}\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2}{\partial z^2}-\epsilon\mu\frac{\partial^2}{\partial t^2}\right)\phi. \tag{24}$$

If the displacement current is not neglected then (9) has the form

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{k}\frac{\partial}{\partial z}[p+\tfrac{1}{2}\mu(\mathbf{H}_0+\mathbf{H})^2+\epsilon\psi]+\epsilon\mu\mathbf{k}[H_0^2-H_{0z}^2]\frac{\partial\phi}{\partial t}$$
$$+\epsilon\mu H_{0z}[\mathbf{k}\times(\mathbf{k}\times\mathbf{H}_0)]\frac{\partial\phi}{\partial t}+\mu H_{0z}\frac{\partial \mathbf{H}}{\partial z}+\lambda\frac{\partial^2\mathbf{v}}{\partial z^2}. \tag{25}$$

$\mathbf{v}$ is perpendicular to $\mathbf{k}$ and thus from (24)

$$\rho\frac{\partial \mathbf{v}}{\partial t} = \lambda\frac{\partial^2\mathbf{v}}{\partial z^2}+\epsilon\mu H_{0z}[\mathbf{k}\times(\mathbf{k}\times\mathbf{H}_0)]\frac{\partial\phi}{\partial t}+\mu H_{0z}\frac{\partial \mathbf{H}}{\partial z}, \tag{26}$$

$$\frac{\partial}{\partial z}[p+\tfrac{1}{2}\mu(\mathbf{H}_0+\mathbf{H})^2+\epsilon\psi] = \epsilon\mu[H_0^2-H_{0z}^2]\frac{\partial\phi}{\partial t}. \tag{27}$$

The velocity and magnetic field components are now determined in terms of ϕ from (25) and (26) and the pressure is then given by (27).

If it is assumed that ϕ is proportional to $\exp i(\omega t+kz)$ then (24) becomes

$$(i\omega+\nu k^2)\left\{\frac{1}{\sigma\mu}(k^2-\epsilon\mu\omega^2)+i\omega\right\} = \frac{\mu H_{0z}^2}{\rho}(\epsilon\mu\omega^2-k^2). \tag{28}$$

For $\nu = 0$, (28) reduces to the dispersion equation obtained by Baños for the p-modes of an inviscid fluid and with suitable changes of notation is identical with the corresponding equation obtained for magneto-elastic waves. For plane waves $\mathbf{v}$ and $\mathbf{H}$ are parallel and thus equation (26) shows that $\mathbf{v}$ is parallel to $\mathbf{k}\times(\mathbf{k}\times\mathbf{H}_0)$; this is precisely Baños's definition of the p-mode solution and the other field components are identical with the ones obtained by Baños for the linearized equations. Thus the p-mode solutions of the linearized equations are possible solutions of the general magnetohydrodynamic equations. If $\mathbf{H}_0$ and $\mathbf{k}$ are parallel, the appropriate form for $\mathbf{E}$ is then $\phi\mathbf{a}\times\mathbf{H}_0$ and it may then be verified that in this case the p- and v-modes are indistinguishable. For $\mathbf{H}_0$ and $\mathbf{k}$ parallel it is possible to obtain some additional exact solutions of the equations and this point will be considered in more detail in the following section.

The fact that the present method of generating p-mode solutions for plane waves is identical with Baños's is a particular example of a general result (Williams 1960) that the general solution of the linearized equations can be expressed in terms of two independent classes of solutions. One class is generated by a stream function and the other by a one component electric vector potential. For plane waves the first class becomes the v-mode solution and the other the p-mode solution.

3. Wave fronts perpendicular to external fields

The problem considered in the first part of the present section is of a more general nature than that of the previous sections in that we investigate the deviations from the uniform state $\mathbf{E} = \mathbf{H} = 0$ of a horizontally stratified liquid. It is also assumed that the permanent magnetic field is vertical. The propagation of small amplitude disturbances in an incompressible fluid of this type has been examined by Ferraro (1954); the corresponding problem for a compressible fluid has been solved by Ferraro & Plumpton (1958). Ferraro has shown that for an incompressible fluid there will be propagated small amplitude disturbances which are essentially Alfvén waves with a variable velocity. It will now be shown that the exact equations also possess solutions of this type.

The undisturbed pressure and density will be denoted by p_0 and ρ_0, respectively, and the deviations from these quantities by p_1 and ρ_1. Equation (3) now becomes

$$(\rho_0+\rho_1)\frac{d\mathbf{v}}{dt} = -\operatorname{grad}(p_0+p_1)+\rho_e\operatorname{div}\mathbf{E}+\mu\mathbf{J}\times(\mathbf{H}_0+\mathbf{H})-(\rho_0+\rho_1)g\mathbf{k}, \tag{29}$$

where g is the acceleration due to gravity. The static equilibrium condition is

$$\frac{dp_0}{dz} = -\rho_0 g, \tag{30}$$

In order to simplify the analysis it has been assumed that the fluid is inviscid; this assumption is not actually essential and the analysis can in fact be carried out for a varying coefficient of viscosity. One solution of equations (6)–(8) is $v_z = H_z = \rho_1 = 0$, and we consider solutions of the equations which satisfy this condition.

We now attempt to obtain a particular solution of the equations with $\mathbf{v}$ and $\mathbf{H}$ parallel; this assumption is made because the solution obtained by Ferraro satisfies it and also it enables the effect of displacement currents to be included. Equations (2) and (4) now show that

$$\left(1+\frac{\epsilon}{\sigma}\frac{\partial}{\partial t}\right)\operatorname{div}\mathbf{E} = 0, \tag{31}$$

and a possible class of solutions of (31) will be $\operatorname{div}\mathbf{E} = 0 = E_z$. From (1), (2) and (4) it may be shown that

$$\left[\mu\frac{\partial}{\partial t}-\frac{1}{\sigma}\left(\frac{\partial^2}{\partial z^2}-\epsilon\mu\frac{\partial^2}{\partial t^2}\right)\right]\mathbf{J} = -\mu\left[\frac{\partial^2}{\partial z^2}-\epsilon\mu\frac{\partial^2}{\partial t^2}\right]\mathbf{v}\times\mathbf{H}_0, \tag{32}$$

and (29) and (32) then give

$$\left[\mu\frac{\partial}{\partial t}-\frac{1}{\sigma}\left(\frac{\partial^2}{\partial z^2}-\epsilon\mu\frac{\partial^2}{\partial t^2}\right)\right]\rho_0\frac{\partial\mathbf{v}}{\partial t}-\mu^2H_0^2\left(\frac{\partial^2}{\partial z^2}-\epsilon\mu\frac{\partial^2}{\partial t^2}\right)\mathbf{v} = -\left[\mu\frac{\partial}{\partial t}-\frac{1}{\sigma}\left(\frac{\partial^2}{\partial z^2}-\epsilon\mu\frac{\partial^2}{\partial t^2}\right)\right]\operatorname{grad}P, \tag{33}$$

where $P = p+\frac{1}{2}\mu H^2+\epsilon\psi$ and ψ is defined as in (18). Since $v_z = 0$ and p is a function of z and t only we have that the right-hand side of (33) vanishes and hence

$$\left\{\mu\rho_0\frac{\partial^2}{\partial t^2}-\left(\frac{\partial^2}{\partial z^2}-\epsilon\mu\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho_0}{\sigma}\frac{\partial}{\partial t}+\mu^2H_0^2\right)\right\}\mathbf{v} = 0. \tag{34}$$

$\mathbf{v}$ will have the form $\phi\mathbf{a}\times\mathbf{k}$, where ϕ is a scalar solution of (34) and $\mathbf{a}$ an arbitrary constant vector; the other components may then be expressed in terms of ϕ.

For $\epsilon = \sigma^{-1} = 0$ equation (34) reduces to the Alfvén wave equation with a variable velocity $(\mu H_0^2/\rho_0)^{\frac{1}{2}}$ and is the equation obtained by Ferraro by linearizing the equations. Thus we see that for this class of problem, also, there exist solutions of the linearized equations which also represent possible solutions of the general equations.

The above analysis has been concerned more with particular forced solutions of the general equations than with the solution of particular boundary-value problems. It is also of interest, however, to examine the possibility of obtaining exact solutions for initial value problems where the wave fronts are perpendicular to the external magnetic field. The fluid is assumed to occupy an unbounded region of space and the initial density is assumed to be uniform; equation (7) then shows that the density will always be uniform. Equation (3) becomes

$$\rho\left\{\frac{\partial}{\partial t}+v_z(t)\frac{\partial}{\partial z}\right\}\mathbf{v} = -\operatorname{grad}\left[p+\tfrac{1}{2}\mu(\mathbf{H}_0+\mathbf{H})^2\right]+\mu H_{0z}\frac{\partial\mathbf{H}}{\partial z}+\lambda\frac{\partial^2\mathbf{v}}{\partial z^2}. \tag{35}$$

From equations (6) and (8) it is seen that the gradient term in (35) vanishes in an unbounded region and hence that $\partial v_z/\partial t = 0$ and v_z is thus constant. From (2), (4) and (8)

$$-\frac{1}{\sigma}\frac{\partial^2\mathbf{H}}{\partial z^2} = -\mu\frac{\partial\mathbf{H}}{\partial t}+\mu\left\{H_{0z}\frac{\partial\mathbf{v}}{\partial z}-v_z\frac{\partial\mathbf{H}}{\partial z}\right\}, \tag{36}$$

where the displacement current has been neglected in deriving (36). Equation (35) now becomes

$$\rho\left\{\frac{\partial}{\partial t}+v_z\frac{\partial}{\partial z}-\nu\frac{\partial^2}{\partial z^2}\right\}\mathbf{v} = \mu H_{0z}\frac{\partial \mathbf{H}}{\partial z}. \tag{37}$$

Eliminating **H** from (36) and (37) gives

$$\rho\left\{\frac{\partial}{dt}+v_z\frac{\partial}{\partial z}-\nu\frac{\partial^2}{\partial z^2}\right\}\left\{v_z\frac{\partial}{\partial z}+\frac{\partial}{\partial t}-\frac{1}{\sigma\mu}\frac{\partial^2}{\partial z^2}\right\}\mathbf{v} = \mu H_{0z}^2\frac{\partial^2 \mathbf{v}}{\partial z^2}. \tag{38}$$

Equation (38) is non-linear but the non-linear effects are due to the constant term v_z and thus the equation is amenable to exact solution. For $\nu = 0$ this equation (without the v_z) was obtained by Roberts (1955) by linearizing the equations of motion and employed by him to solve an initial value problem with $v_z = 0$. The above analysis thus shows that Roberts's solution in fact represents an exact solution of the magnetohydrodynamic equations neglecting displacement current.

In the following section the above analysis will be applied to obtain the magnetohydrodynamic analogues of some exact solutions of the Navier–Stokes equations.

4. Viscous flow problems

One of the classical exact solutions of the Navier–Stokes equations is that for the Rayleigh problem. This problem is essentially the solution of the equations of motion for a viscous incompressible fluid outside an infinite flat plate which is suddenly moved parallel to its length with a velocity U. The corresponding problem when the plate is perfectly conducting and there is a uniform magnetic field present perpendicular to the plate has been solved by Ludford (1959) and Chang & Yen (1959). The plate is assumed to occupy the plane $z = 0$ and to be moved parallel to itself with velocity U in the x-direction. Clearly this problem is of the type considered in §3 and v_x will satisfy (12). The solution obtained by the above authors, by using Laplace transformations, is rather complicated but takes on a considerably simpler form for $\nu = (\mu\sigma)^{-1}$. It is also of interest to note that, for a perfectly conducting fluid, the solution may be obtained from that of a completely different type of initial value problem solved by Roberts (1955).

It is of mathematical interest to note that equation (38) may be used to solve the magnetohydrodynamic Rayleigh problem with constant suction on the plate. This problem has been solved in the absence of a magnetic field by Hasimoto (1956).

The Rayleigh problem for a semi-infinite plate may also be reduced to the solution of a simple linear equation. We assume that the plate occupies the region $z = 0, y \geqslant 0$, for all x; it is then easily verified that a solution of the equations of motion is possible with $v_y = v_z = 0$, and v_x, a function of y, z and t, satisfying

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)v_x = V^2\frac{\partial^2 v_x}{\partial z^2}. \tag{39}$$

$\eta = (\mu\sigma)^{-1}$, ∇^2 is the two-dimensional Laplacian operator and V the Alfvén velocity. The only component of $\mathbf{H}$ is H_x and equations (9) and (10) become

$$H_0\left(\nu\nabla^2-\frac{\partial}{\partial t}\right)v_x+V^2\frac{\partial H_x}{\partial z}=0, \tag{40}$$

$$\left(\eta\nabla^2-\frac{\partial}{\partial t}\right)H_x+H_0\frac{\partial v_x}{\partial z}=0. \tag{41}$$

The electromagnetic boundary conditions are that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ and the normal components of $\mu\mathbf{H}$ are continuous at the plate. It can be shown that the boundary conditions on the plate reduce to $v_x = U$ and $\eta\,\partial H_x/\partial z = 0$. For infinitely conducting fluid the second boundary condition is satisfied identically and in this case the only condition is $v_x = U$. The general boundary-value problem is rather complicated; the Laplace transform method is clearly the appropriate one to employ. The problem is then clearly of the Wiener–Hopf type, the transform variable p will, however, occur in the Wiener–Hopf factorization and the resulting general solution will be extremely complicated. It will, however, be shown that for $\nu = \eta = k_1$, the boundary-value problem reduces to one of a type already solved.

In this case equation (39) becomes

$$\left(k_1\nabla^2-\frac{\partial}{\partial t}-V\frac{\partial}{\partial z}\right)\left(k_1\nabla^2-\frac{\partial}{\partial t}+V\frac{\partial}{\partial z}\right)v_x=0. \tag{42}$$

Thus, from (42), we see that

$$v_x = e^{Vz/2k_1}f(y,z,t)+e^{-Vz/2k_1}g(y,z,t) \tag{43}$$

where f and g are independent solutions of

$$\left(k_1\nabla^2-\frac{\partial}{\partial t}-\frac{V^2}{4k_1}\right)a=0.$$

Equation (40) now gives

$$H_x=-\frac{H_0}{V}\{fe^{Vz/2k_1}-ge^{-Vz/2k_1}\}. \tag{44}$$

Equations (43) and (44) show that the boundary conditions on the half-plane become

$$f+g=U \quad\text{and}\quad \frac{\partial}{\partial z}(f-g)+\frac{V}{2k}(f+g)=0.$$

If ϕ and ψ are defined by

$$\phi=f+g \quad\text{and}\quad \psi=g-f,$$

then the boundary conditions on ϕ and ψ on the half plane are

$$\phi=U \quad\text{and}\quad \partial\psi/\partial z=UV/2k_1.$$

The boundary-value problems for ϕ and ψ are of a well-known type and may be deduced from the solution of the ordinary Rayleigh problem for a half-plane solved by Howarth (1950). The best method of effecting a solution is by means of the Laplace transform. If $\bar{u}(y,z,p)$ denotes the Laplace transform of $u(y,z,t)$, then

$$\bar{u}=\int_0^\infty e^{-pt}u\,dt.$$

We thus require solutions $\bar{\phi}$, $\bar{\psi}$ of

$$\left\{k_1 \nabla^2 - \left(p + \frac{V^2}{4k_1}\right)\right\} \bar{u} = 0, \tag{45}$$

such that $\bar{\phi} = U/p$ and $\partial\bar{\psi}/\partial z = UV/2k_1 p$ on the half-plane. For $V = 0$ Howarth has obtained a solution $\bar{\phi}_H$ of (45) such that $\bar{\phi}_H = U/p$ on the half plane. This solution is defined by

$$\bar{\phi}_H = \frac{U}{2p}\{2\cosh qz - e^{-qz}\operatorname{erf} q^{\frac{1}{2}}(\eta - \xi) - e^{qz}\operatorname{erf} q^{\frac{1}{2}}(\eta + \xi)\}, \tag{46}$$

where $q^2 = p/k_1$ and $y = \xi^2 - \eta^2$ and $z = 2\xi\eta$.

A solution to the present boundary-value problem for $\bar{\phi}$ is thus

$$\bar{\phi} = \left(p + \frac{V^2}{4k_1}\right)\frac{1}{p}\bar{\phi}_H(y, z, p + V^2/4k_1)$$

and hence

$$\phi = e^{-V^2 t/4k_1}\phi_H(y, z, t) + \frac{V^2}{4k_1}\int_0^t e^{-V^2 w/4k_1}\phi_H(y, z, w)\,dw. \tag{47}$$

By a method similar to that of Howarth's it is seen that

$$\bar{\psi} = \frac{UV}{4k_1 p\beta}\{2\sinh\beta z - e^{\beta z}\operatorname{erf}\beta^{\frac{1}{2}}(\eta + \xi) + e^{-\beta z}\operatorname{erf}\beta^{\frac{1}{2}}(\eta - \xi)\}, \tag{48}$$

where $$k_1\beta^2 = p + V^2/4k_1.$$

The inverses of $\bar{\phi}$ and $\bar{\psi}$ may be expressed in terms of integrals of functions obtained by Howarth, which are themselves infinite integrals involving modified Hankel functions. The actual forms of ϕ and ψ are extremely complicated and will therefore not be considered. It is of some interest to examine the behaviour of the solution for large and small values of t; for these cases it is possible to obtain reasonable simple forms for the solution. A quantity which is of physical interest is the skin friction τ at the wall and we shall consider the limiting forms of τ, keeping y fixed, for large and small t. We have that

$$\tau = \left\{\mu H_0 H_x + \rho_0 k_1 \frac{\partial v_x}{\partial z}\right\}_{\text{wall}} = \rho_0 k_1 \left\{\frac{\partial \phi}{\partial z} + \frac{V\psi}{2k_1}\right\}_{\text{wall}}.$$

On the wall

$$\frac{\partial\phi}{\partial z} = e^{-V^2 t/4k_1}\left(\frac{\partial\phi_H}{\partial z}\right)_{\text{wall}} + \left\{\frac{\partial\bar{\phi}_H}{\partial z}(y, z, V/2k_1)\right\}_{\text{wall}} - \frac{V^2}{4k_1}\int_t^\infty e^{-V^2 w/4k_1}\left(\frac{\partial\phi_H}{\partial z}\right)_{\text{wall}} dw \tag{49}$$

For large t, $$\left(\frac{\partial\phi_H}{\partial z}\right)_{\text{wall}} \sim 0{\cdot}4604U(k_1 t y^2)^{-\frac{1}{4}}$$

and thus

$$\frac{1}{U}\left(\frac{\partial\phi_H}{\partial z}\right)_{\text{wall}} \sim 0{\cdot}9208\,(k_1 t y^2)^{-\frac{1}{4}}\,e^{-V^2 t/4k_1} - \left\{\frac{V}{2k_1}\operatorname{erf}\sqrt{\frac{Vy}{2k_1}} + \sqrt{\frac{V}{2k_1 y}}\,e^{-Vy/2k_1}\right\}. \tag{50}$$

One clearly expects that $\partial\phi/\partial z$ will tend to $\partial\phi_H/\partial z$ as $V \to 0$ but equation (50) shows that the asymptotic form of $\partial\phi/\partial z$ is twice that of $\partial\phi_H/\partial z$ in the limiting case of small V. This apparent discrepancy is due to the fact that for V small the integral in (49) may not be estimated by normal methods and a method developed by Clemmow (1950) must be employed. It can be shown that the appropriate form of the first term in (50) is

$$\frac{0{\cdot}4604}{(k_1ty^2)^{\frac{1}{4}}}\left\{e^{-V^2t/4k_1}+\left(\frac{V^2t}{4k_1}\right)^{\frac{1}{4}}\int_{V^2t/4k_1}^{\infty}\frac{e^{-w}}{w^{\frac{1}{4}}}dw\right\}. \tag{51}$$

Equation (51) exhibits the appropriate limiting behaviour for $V=0$ and the integral is an incomplete gamma function. For small t it can be shown that

$$\left(\frac{\partial\phi}{\partial z}\right)_{\text{wall}}\sim\frac{U}{\sqrt{\pi k_1 t}}\left\{1+\frac{V^2t}{4k_1}\right\}.$$

It is easily shown by transform techniques that

$$\psi=-U\operatorname{erf}\sqrt{\frac{Vy}{2k_1}}+O(t^{-\frac{1}{2}})$$

for large t, and

$$\psi=-\frac{UVt^{\frac{1}{2}}}{\pi k_1}+O(t)$$

for small t. Thus we finally obtain

$$\left.\begin{aligned}\tau&=-\rho_0k_1U\left\{\frac{2V}{k_1}\operatorname{erf}\sqrt{\frac{Vy}{2k_1}}+\sqrt{\frac{V}{2k_1y}}\exp\left(-\frac{Vy}{2k_1}\right)-\frac{0{\cdot}9208}{(y^2k,t)^{\frac{1}{4}}}\exp\left(-\frac{V^2t}{4k_1}\right)\right\}\quad\text{as}\quad t\to\infty;\\ \tau&=\frac{\rho_0k_1U}{\sqrt{\pi k_1t}}\left\{1-\frac{V^2t}{4k_1}\right\}\quad\text{as}\quad t\to0.\end{aligned}\right\} \tag{52}$$

For small values of t the skin friction is that obtained by Howarth. The first two terms in (52) represent a residual skin friction; there are two contributions, one from the residual magnetic stress and the other from a residual viscous stress. A similar state of affairs occurs in Ludford's work, the only difference being the absence of a residual viscous stress. The third term in (52) is $2\exp(-V^2t/4k_1)$ times the corresponding skin friction obtained by Howarth, and for small V this term should be replaced by equation (51). The difficulties encountered in the expansion of τ for large values of t may be clarified by writing τ as $\rho_0U(k_1/t)^{\frac{1}{2}}F(R,S)$ where $R^2=y^2/k_1t$, $S=V^2t/k_1$. It is now seen that (52) gives the form of τ for small R assuming S large, whilst equation (51) gives the modification necessary for finite values of S, still assuming R large. A similar situation occurs in Ludford's work and since there appears to be a slight discrepancy in his work we shall consider this point very briefly. Ludford obtains

$$\frac{\tau}{\rho U}=V\operatorname{erf}\frac{V\sqrt{t}}{\sqrt{\eta}+\sqrt{\nu}}+\sqrt{\frac{\nu}{\pi t}}\exp\left(-\frac{V^2t}{(\sqrt{\eta}+\sqrt{\nu})^2}\right),$$

and thus for large t

$$\frac{\tau}{\rho U}\sim\sqrt{\frac{\nu}{\pi t}}\left\{2+\sqrt{\frac{\eta}{\nu}}\right\}\exp\left(-\frac{V^2t}{(\sqrt{\eta}+\sqrt{\nu})^2}\right).$$

The skin friction is thus

$$\left\{2+\sqrt{\frac{\eta}{\nu}}\right\}\exp - V^2t/(\sqrt{\eta}+\sqrt{\nu})^2$$

times the corresponding result obtained by Rayleigh and again a discrepancy occurs in the limiting case of small V. This discrepancy is due to the fact that for small V it is not possible to expand the error function asymptotically. Ludford has obtained -1 instead of $\{2+\sqrt{\eta/\nu}\}$.

Clearly for η and ν small we can obtain a solution by perturbation methods expanding in powers of $\eta-\nu$. The process will be fairly complicated but it will be shown that a solution neglecting squares of $(\eta-\nu)$ may be obtained immediately from the solution for $\nu=\eta$. If k_1 is now defined by $\frac{1}{2}(\eta+\nu)$ then (39) shows that equation (42) will still hold if terms of order $(\eta-\nu)^2$ are neglected. With this new definition of k_1 we define V'_x by

$$V'_x = e^{-Vz/2k_1}f+e^{-Vz/2k_1}g,$$

where f and g satisfy the same boundary conditions as above. We also have from (4) that

$$V^2\frac{\partial \mathbf{H}}{\partial z}+H_0\left(k_1\nabla^2-\frac{\partial}{\partial t}\right)V'_x+\tfrac{1}{2}H_0(\nu-\eta)\nabla^2V'_x = 0. \tag{53}$$

Thus, neglecting terms of order $(\eta-\nu)^2$

$$V^2\frac{\partial H}{\partial z}+\frac{H_0V}{2k_1}(f+g)+H_0\frac{\partial}{\partial z}(f-g)+\tfrac{1}{2}VH_0(\nu-\eta)$$
$$\times\left\{\frac{\partial}{\partial z}(f-g)+\frac{V}{2k_1}(f+g)\right\}+\tfrac{1}{2}H_0(\nu-\eta)\frac{\partial}{\partial t}(f+g) = 0.$$

The conditions on f and g show that, to order $(\nu-\eta)^2$, $\partial H/\partial z = 0$ on the wall. Thus V'_x represents a solution of the problem neglecting terms of order $(\eta-\nu)^2$.

For $\eta = 0$ the solution may be obtained from Howarth's by a suitable variation of the parameters.

Another viscous flow problem capable of exact solutions is that when the plane $z=0$ oscillates parallel to itself. For an insulating wall a solution has been given by Kakutani (1959). The solution for an oscillating half-plane may also be deduced immediately from the Laplace transform of the above solution for impulsive motion.

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